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Winding number of fractional Brownian motion

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Abstract

We find the exact winding number distribution of Riemann–Liouville fractional Brownian motion for large times in two dimensions using the propagator of a free particle. The distribution is similar to the Brownian-motion case and is of Cauchy type. In addition we find the winding number distribution of the fractal-time process, i.e. the time-fractional Fokker–Planck equation, in the presence of a finite-size winding center.

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1. Introduction

Fractional diffusion equations are the basic methods for describing a large class of non-equilibrium phenomena which show a power-law mean-square displacement. In the normal case the mean-square displacement is asymptotically linear in time, i.e. $\langle x^2\rangle \sim t$, but for the anomalous cases it has nonlinear behavior. There are many examples showing anomalous displacement, e.g. protein dynamics [1], relaxation processes and reaction kinetics of proteins [2], two-dimensional rotating flows [3], porous glasses [4], intercellular transport [5] and so on. There are many different kinds of fractional processes but only two of them have more applications and so have been studied in more detail: fractional Brownian motion (FBM) [6] and the fractal-time process with the fractional Fokker–Planck equation [7]. Both processes are non-Markovian and have the power-law mean-square displacement $\langle x^2\rangle \sim t^{2H}$, where H is a real number, but they are fundamentally different [8].

The above processes can describe anomalous diffusion of a particle in time or diffusion of a macromolecule; in other words it is possible to look at these processes as models for dynamics of macromolecules. We have a fractional process which describes the diffusion of a polymer in a specific medium. One of the most important characteristics of a polymer is the winding number distribution which is the simplest quantity describing the entanglement of the macromolecule with point-like molecules or molecules with finite size.

As a simple way to define winding number let us see the process as a two-dimensional random walker. A two-dimensional random walker that starts from the neighborhood of a point in the plane tends to follow a path that wraps around that point, the measure of the wrapping is given via the winding angle, which is the angle around the reference point swept out by the walker. Winding angles of paths are of great interest not only from the mathematical point of view but also because of their application in physics of polymers, flux lines in high temperature superconductors and the quantum Hall effect [9–11]. The winding number of Brownian motion was calculated long time ago by Spitzer [12]. He showed that the distribution of the winding number is of Cauchy type; for some other proofs, see [13–17]. Such a calculation is missing for FBM as a continuous generalization of Brownian motion [6]. In this paper, we will find the exact asymptotic distribution function of winding number of two-dimensional FBM in the infinite plane. In addition, we will derive the same quantity for the fractal-time process in the infinite plane with and without the finite-size winding center.

This paper is organized as follows: in the next section following the same method as in [18] we will find a simple formula similar to Spitzer's result for the winding number distribution of FBM. In section 2 we will calculate the same quantity for the fractal-time process; in this case we will also calculate the winding number distribution in the presence of an obstacle which is a good model to describe the entanglement of a polymer in the presence of a finite-size molecule. Finally the last section summarizes our results.

2. Winding number distribution of FBM

Fractional Brownian motion is defined as a continuous stochastic process with zero mean value and $\langle B_H^2(t)\rangle \sim t^{2H}$, with $0 < H < 1$. To calculate the winding number of such a process we will follow [18] which is based on the methods introduced in the papers [12–15] to derive the winding number distribution of Brownian motion. First we need to have the probability of finding the particle at $\mathbf{r}$ at the time t. This probability is calculated for the Riemann–Liouville fractional Brownian motion in [19, 20]; the definition of this process is

$$B_H(t) = D\int_0^t \frac{\mathrm{d}\tau\xi(\tau)}{(t-\tau)^{1/2-H}}, \tag{2.1}$$

where $\xi(\tau)$ is Gaussian, delta correlated noise $\langle\xi(t)\xi(t')\rangle = \delta(t-t')$, and $D = \frac{1}{\Gamma(H+1/2)}$. This process is not Markovian and does not have stationary increments; therefore, it is different from FBM with stationary increments which is more investigated in the mathematical literature. The Green function of this process satisfies the following diffusion equation,

$$\frac{\partial}{\partial t}G_H(\mathbf{r}_1,\mathbf{r}_2,t) - \frac{1}{2}D^2t^{2H-1}\nabla_{\mathbf{r}_1}^2 G_H(\mathbf{r}_1,\mathbf{r}_2,t) = \delta(\mathbf{r}_2-\mathbf{r}_1)\,\delta(t), \tag{2.2}$$

where $\nabla_{\mathbf{r}_1}^2$ is the Laplace operator acting on $\mathbf{r}_1$. The above equation can be considered as the effective differential equation describing the propagator of FBM (as already proposed in [21][1]). Following [18] one can write the solution of the above equation as

$$G_H(r_1,r_2,\theta,t) = \frac{1}{2\pi}\int_0^\infty\int_0^\infty \exp\left(-\frac{D^2t^{2H}\kappa^2}{2}\right)\cos(\mu\theta)J_\mu(\kappa r)J_\mu(\kappa r')\kappa\,\mathrm{d}\kappa\,\mathrm{d}\mu, \tag{2.3}$$

where $J_\mu(s)$ is the Bessel function of the first kind and r_1 and r_2 are the moduli of $\mathbf{r}_1$ and $\mathbf{r}_2$ respectively. It is possible to look at κ as the eigenvalue of the Laplacian then the above equation is just a bilinear expansion over the corresponding eigenfunctions. The above equation is similar, up to the power of t, to the Brownian motion counterpart and can be checked straightly by putting it into equation (2.2).

[1] It seems that this equation is effectively true and gives the right variance, Green's function and the time dependence of the survival probability [20]. The survival probability is the long-time asymptotics of the probability P_t that the FBM does not escape from a fixed interval up to time t.

It is worth mentioning the exact meaning of the Green function, i.e. $G_H(r_1, r_2, \theta, t)$, it is the statistical weight of trajectories of two-dimensional FBM that start at a point $\mathbf{r}_1$ away from the winding center, here origin, and arrive at another point $\mathbf{r}_2$ after time t after winding angle θ around the origin. If we drop the restriction on the winding around the origin then we can substitute the integral on μ by the sum and after calculating the sum we will get the Gaussian distribution function for the trajectories of two-dimensional FBM that start at a point $\mathbf{r}_1$ and arrive at a point $\mathbf{r}_2$ at time t; the variance of this Gaussian function is t^H, see [19, 20]. The reason for including all of the positive μs in equation (2.3) is as follows: we are trying to calculate the winding number distribution in our problem, so technically θ is different from $\theta \pm 2\pi n$, n is an integer, putting sum in equation (2.3) will not consider this difference. However, we should mention that it is also possible to get the true answer by taking the sum and following Kholodenco's method for finding winding number distribution [17]. The other point to mention is: if we also include negative μs in the integral, since $J_\mu(s)$ for small s is singular for $\mu < 0$, we will not get a finite propagator for $r_1 - r_2 \to 0$. Equation (2.3) is also consistent with propagators in [17, 19, 20].

To get winding number distribution we should evaluate integral (2.3) for large times. The integration over κ is truncated at $\kappa^2 \leqslant \frac{2}{D^2 t^{2H}}$, and so we can replace the Bessel function by the first term of its expansion, i.e. $J_\mu(s) \simeq \frac{1}{\Gamma(1+\mu)}\left(\frac{s}{2}\right)^\mu$. The integration over κ gives

$$G_H(r_1, r_2, \theta, t) \simeq \frac{1}{2\pi D^2 t^{2H}} \int_0^\infty (\lambda_H)^\mu \frac{\cos(\mu\theta)}{\Gamma(1+\mu)}\, d\mu, \qquad (2.4)$$

where $\lambda_H = r_1 r_2/(2D^2 t^{2H})$. If we choose $r_2 = \bar{r} t^H + r_1$ then for large t we have $\lambda_H \simeq \frac{r_1 \bar{r}}{2D^2 t^H}$, which goes to zero for large times. If we consider just small λ_H in the integrand then the integral will be dominated by small μ. The first-order approximation gives

$$G_H(r_1, r_2, \theta, t) \simeq \frac{1}{4\pi D t^{2H}} \frac{\ln\left(\frac{1}{\lambda_H}\right)}{\left(\ln\left(\frac{1}{\lambda_H}\right)\right)^2 + \theta^2}. \qquad (2.5)$$

To get the distribution for θ, since λ_H is small we can use $\ln \lambda_H \simeq -\ln t^H$, after renormalization we have a Cauchy-type distribution

$$G\left(x = \frac{\theta}{H \ln t}\right) = \frac{1}{\pi} \frac{1}{1+x^2}, \qquad (2.6)$$

for t goes to infinity. For $H = \frac{1}{2}$ this is the same as Spitzer's result. The same result is accessible for the other generalizations of Brownian motion defined by $B_H(t) = D\int_0^t \frac{d\tau \xi(\tau)}{\tau^{1/2-H}}$, see [19], with the same constant D. This process has the same propagator but has a different autocorrelation [19, 22]. Since we just need the propagator to find winding number distribution, the winding distribution of this process is the same as the Riemann–Liouville fractional Brownian motion.

It seems that the same calculation should be tractable for different boundary conditions if the propagator of FBM could be calculated; of course translational invariance plays a crucial role in the above argument. The boundary condition corresponding to the problem of walker in the presence of finite-size winding center is equivalent to solving equation (2.2) for the cylindrical boundary condition with a zero generating function on the boundary. We have unfortunately not been able to provide a solution for this problem. However it is possible to do the calculation for the fractal-time process, fractional Fokker–Planck equation, which is the main subject of the following section.

3. Winding number distribution of fractional time process

To define the fractional time process or fractal time random walk we will follow the approach of [23]. Consider a continuous time random walk so that the waiting time between two jumps and the length of the jumps come from a special pdf, $\psi(x,t)$. Then one can get the jump length pdf, $\lambda(x)$, and waiting time pdf, $w(t)$, just by integrating $\psi(x,t)$ on t and x respectively. Using $\psi(x,t)$, the fractal time random walk can be described by the following master equation,

$$\eta(x,t) = \int_{-\infty}^{\infty} \mathrm{d}x' \int_0^{\infty} \mathrm{d}t' \eta(x',t')\psi(x-x',t-t') + \delta(x)\,\delta(t), \tag{3.1}$$

where in the process we have taken into account $w(t) \sim t^{-(1+2H)}$, and the jump's length variance is also finite. We are not going to describe all of the interesting properties of this process; the only thing we need for the rest of the paper is that this process has the following fractional Fokker–Planck equation [23]:

$$\frac{\partial}{\partial t} G_H(\mathbf{r}_1,\mathbf{r}_2,t) - D\frac{\partial^{1-2H}}{\partial t^{1-2H}} \nabla^2_{\mathbf{r}_1} G_H(\mathbf{r}_1,\mathbf{r}_2,t) = \delta(\mathbf{r}_2-\mathbf{r}_1)\,\delta(t). \tag{3.2}$$

In the above formula we used the Riemann–Liouville fractional derivative $\frac{\partial^{1-2H} f(t)}{\partial t^{1-2H}} = \frac{1}{\Gamma(2H-1)} \int_0^t \frac{f(\tau)\,\mathrm{d}\tau}{(t-\tau)^{2-2H}}$, with $1/2 < H < 1$, see [24]. The corresponding derivative for $0 < H < 1/2$ can also be defined by adding the ordinary derivative to the fractional one. Using the above equation it is easy to find the winding number distribution of this process in two dimensions in the presence of a disc-like obstacle in the origin which imposes the cylindrical boundary condition for the above propagator. Following the same steps as in [25] we can find the solution for the cylindrical boundary condition with the zero generating function on the boundary. In the Laplace transform space of t, i.e p, we have

$$\nabla^2_{\mathbf{r}_1} G_H(\mathbf{r}_1,\mathbf{r}_2,p) - \kappa G_H(\mathbf{r}_1,\mathbf{r}_2,p) = -\alpha\delta(\mathbf{r}_2-\mathbf{r}_1), \tag{3.3}$$

where $\kappa \simeq p^{2H}$, and $\alpha \simeq p^{2H-1}$. If we now go to the Fourier space of θ, i.e μ, we will have

$$\left(\frac{\partial^2}{\partial r_1^2} + \frac{1}{r_1}\frac{\partial}{\partial r_1}\right) G_H(r_1,r_2,p,\mu) - \left(\kappa^2 + \frac{\mu^2}{r_1^2}\right) G_H(r_1,r_2,p,\mu) = -\frac{\alpha}{r_1}\delta(r_1-r_2). \tag{3.4}$$

The solution for the above equation is well known, see for example [25], and has the following form:

$$G_H(r_1,r_2,p,\mu) \simeq K_\mu(\kappa r_2)\left(\frac{I_\mu(\kappa r_1)K_\mu(\kappa R) - I_\mu(\kappa R)K_\mu(\kappa r_1)}{K_\mu(\kappa R)}\right), \tag{3.5}$$

where we take $r_2 > r_1$, and R is the radius of the disc removed from the plane and the functions K_μ and I_μ are the modified Bessel functions. To get winding number distribution we need to integrate over the radial coordinate of the final points. Returning to the θ and t spaces by inverse Fourier and inverse Laplace transforms we can write

$$P(\theta,t) = \int_{-\infty}^{\infty} \mathrm{d}\mu\, \mathrm{e}^{\mathrm{i}\mu\theta} C(\mu)\left(\frac{I_\mu(t^{-H}r_1)K_\mu(t^{-H}R) - I_\mu(t^{-H}R)K_\mu(t^{-H}r_1)}{K_\mu(t^{-H}R)}\right), \tag{3.6}$$

where $C(\mu) = \int_{r_1}^{\infty} \mathrm{d}r_2 r_2 K_\mu(t^{-H}r_2)$, for r_1 close to the border of the obstacle. To get the inverse Laplace transform we used the steepest descent approximation similar to [25] for large times which is like substituting κ with t^{-H} after integrating on p space[2]. The above equation

[2] For more detail about the inverse Laplace transform see [18] and for the application of steepest descend method for inverse Laplace transform see [27].

is quite the same as equation (2.15) in [25] with just modified κ, so by following Rudnick and Hu [25] one can argue that the integrand vanishes exponentially at $\pm\infty$ for large times. Then it is possible to calculate the integral by transforming the integration over μ into an integral around a closed contour in the complex plane as the standard contour integration. The integral was calculated in [25] for generic small κ, i.e. in the large time limit, and has the following form:

$$P\left(x = \frac{\theta}{H\ln t}\right) = \frac{\pi}{4\cosh^2\left(\frac{\pi x}{2}\right)}. \tag{3.7}$$

We cannot get the limit of zero winding center from the above equation, to get this limit we should go back to the original equation of $P(\theta, t)$ and take the limit $R \to 0$. It is not difficult to see that just the function I_μ in the integral will survive and we will get the similar equation as we found for Riemann–Liouville fractional Brownian motion. It is also possible to get the same result as (3.7) by following [18], which is the same method but with slightly different technicality.

4. Discussion and summary

Using the Cauchy distribution that we found for both FBM and fractional time process it is not difficult to show that $\langle e^{in\theta}\rangle \sim \frac{1}{t^{nH}}$. It is the same as the result for FBM with autocorrelation $\langle B_H(t+T)B_H(t)\rangle = \frac{1}{2}((t+T)^{2H} + t^{2H} - T^{2H})$, which has stationary increments, at large n [26]. We should emphasize that our result is not necessarily true for this kinds of FBM which has different kinds of Fokker–Planck equation. However it seems that the power-law behavior of the generating function of winding angle at large times for large winding numbers is the general property of different kinds of fractional Brownian processes. Finally we think that it is interesting to calculate the same winding distributions for other familiar situations especially those proposed in [18], such as: paths with fixed endpoints with and without obstacle in winding center and paths with glued endpoints. These cases as argued in [18] are more related to the polymer applications.

It is interesting also to calculate the winding number distribution by using other methods, especially the path integral method, and see the connection to fractional quantum mechanics as already has been done for the ordinary Brownian motion in [13, 15]. It is also interesting to verify our formula by numerical studies.

In conclusion we calculated the exact winding number distribution of Riemann–Liouville fractional Brownian motion for the point-like winding center which is the first result in this context. We did more general calculation for the fractal time process with the finite-size winding center and showed that the equation is, up to a parameter H, similar to the Brownian motion winding number distribution. Moreover we showed that at least for large times and large winding numbers the generating function of winding number is power law in the absence of an obstacle.

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